J Stat Phys (2008) 130: 713–725
DOI 10.1007/s10955-007-9467-x

Non-Lifshitz Tails at the Spectral Bottom of Some Random Operators

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Received: 3 July 2007 / Accepted: 12 November 2007 / Published online: 1 December 2007

Abstract In this paper we continue with the investigation of the behavior of the integrated density of states of random operators of the form $H_\omega = -\nabla \rho_\omega \nabla$. In the present work we are interested in its behavior at 0, the bottom of the spectrum of H_ω. We prove that it converges exponentially fast to the integrated density of states of some periodic operator $\overline{H}$. Being periodic, $\overline{H}$ cannot exhibit a Lifshitz behaviour. This result relates to the result of S.M. Kozlov (Russ. Math. Surv. 34(4):168–169, 1979) and improves it.

Keywords Spectral theory · Random operators · Integrated density of states · Lifshitz tails

1 Introduction

Let H_ω, be the self adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by:

$$H_\omega = H(\rho_\omega) = -\nabla \cdot \rho_\omega \cdot \nabla = \sum_{i=1}^{d} \partial_{x_i} \rho_\omega \partial_{x_i}, \tag{1.1}$$

where ρ_ω is a positive and bounded function. H_ω describes a vibrating membrane in random medium, see [1, 24] for the physical interpretations and the motivation of the study.

Let us start by defining the main object of our study: the integrated density of states. For this, we consider Λ a cube of $\mathbb{R}^d$. We note by $H_{\omega,\Lambda}$ the restriction of H_ω to Λ with self-adjoint boundary conditions. As H_ω is elliptic, the resolvent of $H_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $H_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity [22]. We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\text{eigenvalues of } H_{\omega,\Lambda} \le E\}. \tag{1.2}$$

Research partially supported by the Research Unity 01/UR/ 15-01 projects.

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Here $\mathrm{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#A$ is the cardinal of A.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the *integrated density of states* of H_ω (IDS as acronym) in what follows we denote it by $\mathcal{N}$. See [21].

The question we are interested in here concerns the behavior of $\mathcal{N}$ at the bottom of the spectrum of H_ω.

Let us give a brief history of this subject. In 1964, Lifshitz [14] argued that, for a Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2, \alpha > 0$ such that $\mathcal{N}(E)$ satisfies

$$\mathcal{N}(E) \simeq c_1 \exp(-c_2(E - E_0)^{-\alpha}), \quad E \to E_0. \tag{1.3}$$

Here E_0 is the bottom of the spectrum of H_ω. The behavior (1.3) is known as *Lifshitz tails* (for more details see part IV.9.A of [21]), and α is the Lifshitz exponent. Usually such an exponent is of the form $\frac{d}{2}$, where d is the dimension. We notice that the Lifshitz behavior is among the properties expected for random operators.

Lifshitz predicted (1.3) also at fluctuating edges inside the spectrum. We refer to this asymptotics by "*internal Lifshitz tails*".

The principal known results on Lifshitz tails are mainly shown for Schrödinger operators (for continuous and discrete cases). See [6–8, 10, 13, 21, 23] where some of them use the Donsker and Varadhan technic [4, 5].

Lifshitz tails for an operator of type (1.1), was the subject of previous works [15–18], where we get the behavior of $\mathcal{N}$ at the internal band edges of the spectrum of (1.1). It was a Lifshitz behavior ($\mathcal{N}$ decreases exponentially fast at the internal edges).

For classical random Schrödinger operator it is known that the bottom of the spectrum is a fluctuating edge [6, 19]. In the present situation, for the spectrum minimum, it should be noted that 0 is not a fluctuating edge of the spectrum [24]. It belongs to the spectrum of H_ω independently of the choice of the random variables in $\rho_\omega(x)$. As ρ_ω is bounded, using a variational formula and a Weyl type asymptotics one gets that there exists $C > 1$ such that

$$\frac{E^{d/2}}{C} \le \mathcal{N}(E) \le C E^{d/2}.$$

This yields that $\mathcal{N}$ can not decrease faster than $E^{\frac{d}{2}}$ at 0.

In [20] elliptic operator in the divergence form on a random strip is considered, and it is proved that the integrated density of states of the relevant operator exhibits the Lifshitz behavior at the bottom of the spectrum. We notice that in [20] Lifshitz tails at the bottom of the spectrum is a consequence of the geometry of the domain.

For $d = 2$, under some assumption on ρ_ω, and using the Laplace transformation of $\mathcal{N}$, the fundamental solution of the heat equation as well as Tauberian Theorem, Kozlov [12] gets the behavior of $\mathcal{N}$ with an error order of $E^{d/2}$.

In the present work using periodic approximations and probabilistic arguments we compare the behavior of the IDS of H_ω to that of some periodic operator $\overline{H}$ with an exponential precision.

1.1 The Model

Consider the operator

$$H_\omega = -\nabla \rho_\omega \nabla, \tag{1.4}$$

where ρ_ω is a bounded, $\mathbb{Z}^d$-ergodic random field such that there exists some constant $\rho_* > 1$, satisfying

$$\rho_*^{-1} \le \rho_\omega \le \rho_*. \tag{1.5}$$

We assume that ρ_ω is of Anderson type i.e. it has the form

$$\rho_\omega(x) = \rho^+(x) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma \rho^0(x-\gamma), \tag{1.6}$$

where

- ρ^+ is a-$\mathbb{Z}^d$-periodic measurable function,
- ρ^0 is a compactly supported measurable function,
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are non trivial, i.i.d. non degenerate random variables.

Let $\mathcal{H}(\rho_\omega)$ be the quadratic form defined as follow: for $u \in H^1(\mathbb{R}^d) = \mathcal{D}(\mathcal{H}(\rho_\omega))$

$$\mathcal{H}(\rho_\omega)[u,u] = \int_{\mathbb{R}^d} \rho_\omega(x)\nabla u(x)\overline{\nabla u(x)}dx.$$

$\mathcal{H}(\rho_\omega)$ is a symmetrical, closed and positive quadratic form. H_ω given by (1.1) is defined to be the Friedrichs extension of $\mathcal{H}(\rho_\omega)$ [22].

The choice of our model ensures that H_ω is a measurable family of self-adjoint operators and ergodic [6, 21]. Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \mathbb{Z}^d$ we have

$$\tau_\gamma H_\omega \tau_{-\gamma} = H_{\tau_\gamma\omega}.$$

According to [6, 21], we know that there exists Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of H_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of H_ω, then $\Sigma_{pp} = \sigma_{pp}$, $\Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one. It should be noted that the bottom of the spectrum of H_ω is 0, and this is independent of the choice of $\rho_\omega(x)$.

1.2 The Result

We shall prove

Theorem 1.1 *There exists $\tau > 0$, $0 < \alpha \le d/(4(d+1))$ and $C > 1$ such that when $E \to 0^+$ we have,*

$$\overline{\mathcal{N}}(0) - Ce^{-E^{-\tau}} \le \mathcal{N}(E) \le \overline{\mathcal{N}}(E + E^\alpha) + Ce^{-E^{-\tau}}, \tag{1.7}$$

where $\overline{\mathcal{N}}$ is the integrated density of states of the following periodic operator

$$\overline{H} = -\nabla\overline{\rho}\nabla; \tag{1.8}$$

and $\overline{\rho} = \mathbb{E}(\rho_\omega)$.

Remark 1.2 (1) Notice that the improvement over Kozlov's result essentially consists in the estimate of the remainder term and the exponential precision and the dimension d can be different of 2 in the present study.

(2) We do not believe this estimate to be optimal: namely, we expect the exponent α to be larger than the one we found in the present study.

2 The Periodic Approximations

Pick $n \in \mathbb{N}\backslash\{0\} = \{1, 2, 3, \ldots\}$ and define the following periodic operator

$$H_\omega^n = -\nabla \rho_\omega^n \nabla,$$

where,

$$\rho_\omega^n = \rho^+ + \rho_\omega^{0,n} = \rho^+(x) + \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta),$$

and

$$\Lambda_n = \left\{ x \in \mathbb{R}^d; \forall 1 \leq j \leq d,\ -\frac{2n+1}{2} < x_j \leq \frac{2n+1}{2} \right\}.$$

For ω fixed and $n \in \mathbb{N}\backslash\{0\}$, H_ω^n is a $(2n+1)\mathbb{Z}^d$-periodic self-adjoint Schrödinger operator.

Let $\overline{\omega} = \mathbb{E}(\omega_0)$. We set

$$\overline{\rho^n} = \rho^+ + \overline{\rho^{0,n}} = \rho^+ + \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \overline{\omega} \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta),$$

and

$$\overline{H}^n = -\nabla \overline{\rho^n} \nabla.$$

2.1 Some Floquet Theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references of this material are in [22].

Let the torus $\mathbb{T}_{2n+1}^* = \mathbb{R}^d / \{\frac{2\pi}{(2n+1)} \mathbb{Z}^d\}$. We define $\mathcal{H}_n$ by

$$\begin{aligned} \mathcal{H}_n = \{& u(x,\theta) \in L^2_{loc}(\mathbb{R}^d) \otimes L^2(\mathbb{T}_{2n+1}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d \times \mathbb{T}_{2n+1}^* \times (2n+1)\mathbb{Z}^d; \\ & u(x+\gamma,\theta) = e^{i\gamma\theta} u(x,\theta)\}. \end{aligned}$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}_n$ such that H_ω^n admits the following Floquet decomposition [22]:

$$U H_\omega^n U^* = \int_{\mathbb{T}_{2n+1}^*}^{\oplus} H_\omega^n(\theta) d\theta.$$

Here $H_\omega^n(\theta)$ is the self adjoint operator on $\mathcal{H}_{n,\theta}$ defined as the operator H_ω^n acting on $\mathcal{H}_{n,\theta}^1$ with

$$\mathcal{H}_{n,\theta} = \{u \in L^2_{loc}(\mathbb{R}^d); \forall \gamma \in (2n+1)\mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta} u(x)\},$$

and

$$\mathcal{H}^1_{n,\theta} = \{u \in \mathcal{H}_{n,\theta};\ \partial_x^\alpha u \in \mathcal{H}_{n,\theta};\ |\alpha| = 1\}.$$

We define $H^n(\theta)$ by the same meaning.

As H^n_ω is elliptic, we know that, $H^n_\omega(\theta)$ has a compact resolvent; hence its spectrum is discrete [22]. We denote its eigenvalues, called Floquet eigenvalues of $H^n_\omega(\theta)$, by

$$E_0(n,\omega,\theta) \le E_1(n,\omega,\theta) \le \cdots \le E_k(n,\omega,\theta) \le \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_k)_{k\in\mathbb{N}}$. The functions $(\theta \to E_k(n,\omega,\theta))_{k\in\mathbb{N}}$ are Lifshitz-continuous, and we have

$$E_k(n,\omega,\theta) \to +\infty \quad \text{as } k \to +\infty \text{ uniformly in } \theta.$$

The spectrum $\sigma(H^n_\omega)$ of H^n_ω has a band structure, i.e.

$$\sigma(H^n_\omega) = \bigcup_{k\in\mathbb{N}} E_k(n,\omega,\mathbb{T}^*_{2n+1}).$$

Let $\mathcal{N}^n_\omega$ be the integrated density of states of H^n_ω; it satisfies

$$\mathcal{N}^n_\omega(E) = \sum_{k\in\mathbb{N}} \frac{1}{(2\pi)^d} \int_{\{\theta\in\mathbb{T}^*_{2n+1};\ E_k(n,\omega,\theta)\le E\}} d\theta = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^*_{2n+1}} \mathcal{V}(H^n_\omega(\theta), E)d\theta. \tag{2.9}$$

Here $\mathcal{V}(B,E)$ is the number of eigenvalues of B less or equal to E. Let $d\mathcal{N}^n_\omega$ be the derivative of $\mathcal{N}^n_\omega$ in the distribution sense. As $\mathcal{N}^n_\omega$ is increasing, $d\mathcal{N}^n_\omega$ is a positive measure; it is the density of states of H^n_ω. We denote by $d\mathcal{N}$ the density of states of H_ω. For all $\varphi \in C_0^\infty(\mathbb{R})$, $d\mathcal{N}^n_\omega$ verifies [10],

$$\langle \varphi, d\mathcal{N}^n_\omega\rangle = \frac{1}{(2\pi)^d} \int_{\theta\in\mathbb{T}^*_{2n+1}} \mathrm{tr}_{\mathcal{H}_{n,\theta}}(\varphi(H^n_\omega(\theta)))d\theta = \frac{1}{\mathrm{vol}(\Lambda_k)} \mathrm{tr}(\chi_{\Lambda_k}\varphi(H^n_\omega)\chi_{\Lambda_k}), \tag{2.10}$$

where for $\Lambda \subset \mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\mathrm{tr}(A)$ is the trace of A (we index by $\mathcal{H}_{n,\theta}$ if the trace is taken in $\mathcal{H}_{n,\theta}$). Now, we state a result proven in [16].

Lemma 2.1 [16] *For any $\varphi \in C_0^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{n\to\infty} \mathbb{E}(\langle \varphi, d\mathcal{N}^n_\omega\rangle) = \langle \varphi, d\mathcal{N}\rangle.$$

Moreover, we have that the IDS of H_ω is exponentially well-approximated by the expectation of the IDS of the periodic operators H^n_ω when n is polynomial in ε^{-1}. More precisely we have

Theorem 2.2 [17] *Pick $\eta > 0$ and $I \subset \mathbb{R}$, a compact interval. There exists $\varepsilon_0 > 0$ and $\rho > 0$ such that, for $E \in I$, $\varepsilon \in (0, \varepsilon_0)$ and $n \ge \varepsilon^{-\rho}$, one has*

$$\begin{aligned} &\mathbb{E}(\mathcal{N}^n_\omega(E+\varepsilon/2)) - \mathbb{E}(\mathcal{N}^n_\omega(E-\varepsilon/2)) - e^{-\varepsilon^{-\eta}} \\ &\le \mathcal{N}(E+\varepsilon) - \mathcal{N}(E-\varepsilon) \\ &\le \mathbb{E}(\mathcal{N}^n_\omega(E+2\varepsilon)) - \mathbb{E}(\mathcal{N}^n_\omega(E-2\varepsilon)) + e^{-\varepsilon^{-\eta}}. \end{aligned} \tag{2.11}$$

Remark 2.3 This lemma is proven in [17] (Lemma 3.3) for acoustic operators and in [11] for the Schrödinger case. The proof is based on the Helffer-Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes-Thomas argument) and the properties of Gevrey class functions.

2.2 The Study of the IDS of the Periodic Approximations

As it is mentioned in this section we will study the IDS of the periodic approximations and precisely we will prove an analogous theorem to Theorem 1.1.

Theorem 2.4 *There exists $\alpha > 0$ and a set $\Omega_{n,E,\alpha}$ such that we have,*

$$\overline{\mathcal{N}}^n(E - E^\alpha) - C\mathbb{P}(\Omega_{n,E,\alpha}) \le \mathcal{N}_\omega^n(E) \le \overline{\mathcal{N}}^n(E + E^\alpha) + C\mathbb{P}(\Omega_{n,E,\alpha}), \tag{2.12}$$

where $\overline{\mathcal{N}^n}$ is the IDS of $\overline{H}^n$.

For a vector space $\mathcal{E}$, we note by $\dim(\mathcal{E})$ the dimension of $\mathcal{E}$. We have,

$$\begin{aligned}\mathcal{V}(H_\omega^n(\theta), E) &= \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{H}_{n,\theta}^1, \text{ such that, } \forall u \in \mathcal{E};\ \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\} \\ &= \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{H}_{n,\theta}^1, \text{ such that, } \forall u \in \mathcal{E}; \\ &\qquad \langle (H_\omega^n(\theta) - \overline{H}^n(\theta))u, u\rangle + \langle \overline{H}^n(\theta)u, u\rangle \le E\|u\|^2\}.\end{aligned}$$

Let $\alpha > 0$. We set

$$\mathcal{E}_1^\alpha(\theta) = \{u \in \mathcal{H}_{n,\theta}^1;\ |\langle (H_\omega^n(\theta) - \overline{H}^n(\theta))u, u\rangle| \le E^\alpha\|u\|^2\},$$

and

$$\mathcal{E}_2^\alpha(\theta) = \{u \in \mathcal{H}_{n,\theta}^1;\ |\langle (H_\omega^n(\theta) - \overline{H}^n(\theta))u, u\rangle| \ge E^\alpha\|u\|^2\}.$$

Then we have,

$$\begin{aligned}&\mathcal{V}(H_\omega^n(\theta), E) \\ &\quad \le \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_1^\alpha(\theta), \text{ such that, } \forall u \in \mathcal{E};\ \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\} \\ &\qquad + \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_2^\alpha(\theta), \text{ such that, } \forall u \in \mathcal{E};\ \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\} \\ &\quad \le \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_1^\alpha(\theta), \text{ such that, } \forall u \in \mathcal{E};\ \langle \overline{H}^n(\theta)u, u\rangle \le (E + E^\alpha)\|u\|^2\} \\ &\qquad + \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_2^\alpha(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\}.\end{aligned}$$

So, we get

$$\begin{aligned}&\mathcal{V}(H_\omega^n(\theta), E) \le \mathcal{V}(\overline{H}^n(\theta), (E + E^\alpha)) \\ &\qquad + \sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_2^\alpha(\theta), \text{ such that, } \forall u \in \mathcal{E}; \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\}.\end{aligned} \tag{2.13}$$

Now integrating both sides of (2.13) over $\mathbb{T}_{2n+1}^*$ and taking into account (2.9), we get that

$$\begin{aligned}\mathcal{N}_\omega^n(E) \le \overline{\mathcal{N}}^n((E + E^\alpha)) + \frac{1}{(2\pi)^d}\int_{\mathbb{T}_{2n+1}^*} &\sup\{\dim \mathcal{E};\ \mathcal{E} \subset \mathcal{E}_2^\alpha(\theta), \text{ such that,} \\ &\forall u \in \mathcal{E}; \langle H_\omega^n(\theta)u, u\rangle \le E\|u\|^2\} d\theta,\end{aligned} \tag{2.14}$$

where $\overline{\mathcal{N}}^n$ is the IDS of $\overline{H}^n$.

As,

$$\begin{aligned}&\sup\{\dim\mathcal{E};\ \mathcal{E}\subset\mathcal{E}_2^\alpha(\theta),\text{ such that, }\forall u\in\mathcal{E};\langle H_\omega^n(\theta)u,u\rangle\le E\|u\|^2\}\\&\quad\le\dim\{u;\ \forall u\in\mathcal{H}^1_{n,\theta};\ \langle H_\omega^n(\theta)u,u\rangle\le E\|u\|^2\}.\end{aligned}\tag{2.15}$$

Using (1.5), we get that the right side of (2.15) is bounded by the number of eigenvalues of $-\Delta^n(\theta)$ less than $E\rho_*$ which is it self bounded by Cn^d (C depends only on E). As the volume of $\mathbb{T}^*_{2n+1}$ is $(\frac{2\pi}{(2n+1)})^d$ we get that for some $C>0$ we have

$$\mathbb{E}(\mathcal{N}_\omega^n(E))\le\overline{\mathcal{N}}^n(E+E^\alpha)+C\mathbb{P}(\Omega_{n,E,\alpha}),\tag{2.16}$$

with

$$\Omega_{n,E,\alpha}=\{\omega;\ \exists\theta\in\mathbb{R}^d;\exists u\in\mathcal{E}_2^\alpha(\theta);\ \langle\nabla u,\nabla u\rangle\le E\rho_*\|u\|^2\}.$$

Now let us consider

$$\begin{aligned}&\mathcal{V}(\overline{H}^n(\theta),(E-E^\alpha))\\&\quad=\sup\{\dim\mathcal{E};\ \mathcal{E}\subset\mathcal{H}^1_{n,\theta},\text{ such that, }\forall u\in\mathcal{E};\ \langle\overline{H}^n(\theta)u,u\rangle\le(E-E^\alpha)\|u\|^2\}\\&\quad=\sup\{\dim\mathcal{E};\ \mathcal{E}\subset\mathcal{H}^1_{n,\theta},\text{ such that, }\forall u\in\mathcal{E};\\&\qquad\quad\langle(\overline{H}^n(\theta)-H_\omega^n(\theta))u,u\rangle+\langle H_\omega^n(\theta)u,u\rangle\le(E-E^\alpha)\|u\|^2\}\\&\quad\le\sup\{\dim\mathcal{E};\ \mathcal{E}\subset\mathcal{E}_1^\alpha(\theta),\text{ such that, }\forall u\in\mathcal{E};\ \langle\langle H_\omega^n(\theta)u,u\rangle\le E\|u\|^2\}\\&\qquad+\sup\{\dim\mathcal{E};\ \mathcal{E}\subset\mathcal{E}_2^\alpha(\theta),\text{ such that, }\forall u\in\mathcal{E};\ \langle\overline{H}^n(\theta)u,u\rangle\le(E-E^\alpha)\|u\|^2\}.\end{aligned}$$

Using the same computation carried out from (2.13–2.16), we get that

$$\overline{\mathcal{N}}^n(E-E^\alpha)-C\mathbb{P}(\Omega_{n,E,\alpha})\le\mathcal{N}_\omega^n(E).\tag{2.17}$$

This ends the proof of Theorem 2.4. □

3 The Proof of Theorem 1.1

To prove Theorem 1.1 it suffice to take into account the results of Theorems 2.4 and 2.2 and Lemma 2.1 and estimate $\mathbb{P}(\Omega_{n,E,\alpha})$. It is the purpose of the following lemma. It is a large deviation argument.

Lemma 3.1 *There exists $\tau>0$ such that for E sufficiently small and n large, we have*

$$\mathbb{P}(\Omega_{n,E,\alpha})\le e^{-E^{-\tau}}.$$

Now the proof of Theorem 1.1 is just to take into account Theorem 2.2 and Lemma 3.1

Proof of Lemma 3.1 We prove this Lemma using techniques of [9]. We have $\Omega_{n,E,\alpha}\subset\Omega'_{n,E,\alpha}$, with

$$\begin{aligned}\Omega'_{n,E,\alpha}=\{\omega;\ \exists\theta\in\mathbb{R}^d;\ \exists u\in H^1(\mathbb{R}^d);\ \|u\|_{L^2(\mathbb{R}^d)}=1;\ \|\nabla u\|^2\le E\rho_*;\\ \text{and }|\langle(H_\omega^n(\theta)-\overline{H}^n(\theta))u,u\rangle|\ge E^\alpha\}.\end{aligned}$$

Let us estimate the probability of the latest events. Notice that we asked that

$$|\langle (H_\omega^n(\theta)-\overline{H}^n(\theta))u,u\rangle| = \left|\sum_{i=1}^d \langle (\rho_\omega^{0,n}(\theta)-\overline{\rho^{0,n}}(\theta))\partial_{x_i}u, \partial_{x_i}u\rangle\right| \geq E^\alpha \|u\|^2. \tag{3.18}$$

Let $u \in H^1(\mathbb{R}^d)$, then u can be written using the Floquet decomposition of $-\Delta$ as:

$$u = \sum_{k\in\mathbb{N}} \int_{\mathbb{T}^*_{2n+1}} \chi_k(\theta) w_k(x,\theta) d\theta,$$

where $(w(\cdot,\theta)_k)_{k\geq 0}$ are the Floquet eigenfunctions of $-\Delta_n^\theta$ associated to $(E_k(\theta))_{k\geq 0}$.

By this, for u such that $\langle -\Delta u, u\rangle \leq E\rho_*$ we have:

$$\left(\sum_{k\geq 0} \int_{\mathbb{T}^*_{2n+1}} |E_k(\theta)|^2 |\chi_k(\theta)|^2 d\theta\right) \leq CE^2. \tag{3.19}$$

0 is the bottom of the spectrum of $-\Delta$. It is a simple non-degenerate Floquet eigenvalue [22]. Hence there exists $C > 0$ such that

- For $k \neq 0, \forall \theta \in \mathbb{T}^*_{2n+1}$

$$|E_k(\theta)| \geq 1/C, \tag{3.20}$$

- And $\exists Z = \{\theta_j \in \mathbb{T}^*_{2n+1};\ 1 \leq j \leq n_0\}$ such that $E_0(\theta_j) = 0$.

$$|E_0(\theta)| \geq 1/C \inf_{1\leq j\leq n_0} |\theta - \theta_j|^2. \tag{3.21}$$

Let $(2l+1) = [E^{-1/2+2\rho'}]_\circ \cdot [E^{-\rho'}]_\circ$ and $(2k+1) = [E^{-\eta}]_\circ$, where $\alpha < \rho' < \frac{d}{4(d+1)}$ and $\eta > 0$ such that $(2l+1)\cdot(2k+1) = 2n+1$. Here $[\cdot]_\circ$ denotes the largest odd integer smaller than $\cdot$.

From (3.19–3.21) we get that

$$\sum_{k\geq 1} \int_{\mathbb{T}^*_{2n+1}} |\chi_k(\theta)|^2 d\theta + \sum_{j=1}^{n_0} \int_{|\theta-\theta_j|>\frac{1}{l}} |\chi_0(\theta)|^2 d\theta \leq CE^2 l^2 \leq CE^{2\rho'}. \tag{3.22}$$

Hence we write

$$u = \sum_{j=1}^{n_0} u_j + u^e, \quad \text{where } u_j = \int_{|\theta-\theta_j|\leq\frac{1}{l}} \chi_0(\theta) w_0(\cdot,\theta) d\theta.$$

So, we have

$$\|u^e\|^2 \leq CE^{2\rho'}, \tag{3.23}$$

and

$$\sum_{j=1}^{n_0} \|u_j\|^2 = \|u\|^2 - CE^{2\rho'} = 1 - CE^{2\rho'}.$$

We notice that using the fact that $w(\cdot,\theta) \in \mathcal{H}^1_{n,\theta}$, and (3.23) is based on the choice of l, the localisation length in the quasimomentum θ, we get that

$$\|\nabla u^e\|^2 \le C E^{2\rho'}. \tag{3.24}$$

Now using (3.24) in (3.18), we get that for E small we have

$$\sum_{1\le j,j'\le n_0} |\langle(\rho_\omega^{0,n} - \overline{\rho^{0,n}})\nabla u_j, \nabla u_{j'}\rangle| \ge E^\alpha/4. \tag{3.25}$$

So, for some $1 \le j, j' \le n_0$, one has

$$|\langle(\rho_\omega^{0,n} - \overline{\rho^{0,n}})\nabla u_j, \nabla u_{j'}\rangle| = \left|\sum_{i=1}^{d}\langle(\rho_\omega^{0,n} - \overline{\rho^{0,n}})\partial_{x_i} u_j, \partial_{x_i} u_{j'}\rangle\right| \ge E^\alpha/(2n_0)^2. \tag{3.26}$$

Now we state a lemma based on the Uncertainly principle and proved in [9] (see Lemma 2.1 and Lemma 5.1 in [9]).

Lemma 3.2 [9] *Fix $1 \le j \le n_0$ and $1 \le i \le d$. For $1 \le l' \le l$ there exists $\widetilde{u}_j \in L^2(\mathbb{R}^d)$ such that*

(1) *$\widetilde{u}_j$ is constant on each cube*

$$\Lambda_{\gamma,l'} = \left\{x = (x_1,\dots,x_d);\ \forall 1 \le i \le d, -l' - \frac{1}{2} \le x_i - (2l'+1)\gamma_i < l' + \frac{1}{2}\right\},$$

where $\gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{Z}^d$.

(2) *$\exists C > 0$ depending only on $w_0(\cdot,\theta)$ such that*

$$\|\partial_{x_i} u_j - \widetilde{u}_j \cdot \partial_{x_i}\overline{w}_0(\cdot,\theta_j)\|_{L^2(\mathbb{R}^d)} \le Cl'/l, \tag{3.27}$$

where $\overline{w}_0(\cdot,\theta)$ is the periodic component of $w_0(\cdot,\theta)$ i.e. $w_0(\cdot,\theta) = e^{ix\theta}\overline{w}_0(\cdot,\theta)$.

Let

$$\psi_j^i(x) = \widetilde{u}_j(x)\partial_{x_i}\overline{w}_0(x,\theta_j) = \partial_{x_i}\overline{w}_0(x,\theta_j)\sum_{\beta\in\mathbb{Z}^d}(2l'+1)^{-d/2}a_j(\beta)\mathbf{1}_{(2l'+1)\beta+\Lambda_{0,l'}}.$$

So, for any $1 \le i \le d$ and $1 \le j \le n_0$, $\exists C > 0$ depending only on $w_0(\cdot,\theta)$ such that

$$\|\partial_{x_i} u_j - \psi_j^i\|_{L^2(\mathbb{R}^d)} \le Cl'/l. \tag{3.28}$$

We notice that $\psi_j^i(x) = \widetilde{u}_j(x)\partial_{x_i}\overline{w}_0(x,\theta_j) \in L^2(\mathbb{R}^d)$. Using the periodicity of $\overline{w}_0(x,\theta_j)$ we get,

$$\|\psi_j^i\|^2_{L^2(\mathbb{R}^d)} = \|\widetilde{u}_j(x)\partial_{x_i}\overline{w}_0(x,\theta_j)\|^2_{L^2(\mathbb{R}^d)} = \sum_{\beta\in\mathbb{Z}^d}|a_j(\beta)|^2\int_{\Lambda_{0,0}}|\partial_{x_i}\overline{w}_0(x,\theta_j)|^2dx. \tag{3.29}$$

Then using (3.28) and the fact that $\int_{\Lambda_{0,0}}|\partial_{x_i} w_0(x,\theta_j)|^2dx = \int_{\Lambda_{0,0}}|\partial_{x_i}\overline{w}_0(x,\theta_j)|^2dx$; we get that there exists $C > 0$ such that

$$\sum_{\beta\in\mathbb{Z}^d}|a_j(\beta)|^2 \le C\|\partial_{x_i} u_j\|_{L^2(\mathbb{R}^d)} < +\infty. \tag{3.30}$$

We set $2l'+1=[E^{-1/2+2\rho'}]_\circ$ and $2k'+1=[E^{-\rho'}]_\circ\cdot[E^{-\eta}]_\circ$, for $\alpha<\rho'<\frac{d}{4(d+1)}$ and $\eta>0$ so that $(2n+1)=(2l'+1)\cdot(2k'+1)$. So taking into account (3.26, 3.28) and the choice of l and l', we get

$$\sum_{i=1}^{d}|\langle(\rho_\omega^{0,n}-\overline{\rho^{0,n}})\psi_j^i,\psi_{j'}^i\rangle|\geq E^\alpha/(2n_0)^2-CE^{\rho'}\geq E^\alpha/(4n_0)^2. \tag{3.31}$$

We set

$$\sum_{i=1}^{d}|\langle(\rho_\omega^{0,n}-\overline{\rho^{0,n}})\psi_j^i,\psi_{j'}^i\rangle|=\sum_{1\leq i\leq d}|A_{j,j'}^i|,$$

with

$$A_{j,j'}^i=\langle(\rho_\omega^{0,n}-\overline{\rho^{0,n}})\psi_j^i,\psi_{j'}^i\rangle.$$

We have

$$\begin{aligned}A_{j,j'}^i&=\sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}(2l'+1)^{-d}a_j(\beta)\cdot\overline{a_{j'}}(\beta)\\&\quad\times\int_{(2l'+1)\beta+\Lambda_{0,l'}}(\rho_\omega^{0,n}-\overline{\rho^{0,n}})(x-\gamma)\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot\overline{\partial_{x_i}\overline{w}_0(x,\theta_j')}dx\\&=\sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}a_j(\beta)\cdot\overline{a_{j'}(\beta)}\\&\quad\times\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}}(\rho_\omega^{0,n}-\overline{\rho^{0,n}})(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\\&\quad\times\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx.\end{aligned}\tag{3.32}$$

As ρ_ω^n is $(2n+1)\mathbb{Z}^d$-periodic and $(2l'+1)(2k'+1)=(2n+1)$ we get that

$$\begin{aligned}A_{j,j'}^i&=\sum_{\beta\in\mathbb{Z}^d_{2k'+1}}\sum_{\beta'\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}a_j(\beta+(2k'+1)\beta')\cdot\overline{a_{j'}(\beta+(2k'+1)\beta')}\\&\quad\times\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}}(\rho_\omega^{0,n}-\overline{\rho^{0,n}})(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\\&\quad\times\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx.\end{aligned}\tag{3.33}$$

Using the expression of ρ_ω we get that.

$$\begin{aligned}A_{j,j'}^i&=\sum_{\beta\in\mathbb{Z}^d_{2k'+1}}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}\sum_{\beta'\in\mathbb{Z}^d}(\omega_\gamma-\overline{\omega})a_j(\beta+(2k'+1)\beta')\overline{a_{j'}(\beta+(2k'+1)\beta')}\\&\quad\times\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}}\rho^0(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx.\end{aligned}\tag{3.34}$$

We set

$$B_{j,j'}^i(\beta)=\sum_{\beta'\in\mathbb{Z}^d}a_j(\beta+(2k'+1)\beta')\cdot\overline{a_{j'}(\beta+(2k'+1)\beta')}.\tag{3.35}$$

Then we get that

$$A^i_{j,j'} = (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (\omega_\gamma - \overline{\omega}) \Bigg(\sum_{\beta\in\mathbb{Z}_{2k'+1}} B^i_{j,j'}(\beta) \times \int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot \overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx \Bigg)$$

$$= (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} \Bigg[\sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\omega_{\gamma+(2l'+1)\gamma'} - \overline{\omega}) \Bigg(\sum_{\beta\in\mathbb{Z}_{2k'+1}} B^i_{j,j'}(\beta) \times \int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)(\beta-\gamma'))\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot \overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx \Bigg) \Bigg]. \quad (3.36)$$

We set

$$C^i_{j,j'}(\gamma,\gamma') = \sum_{\beta\in\mathbb{Z}^d_{2k'+1}} B^i_{j,j'}(\beta) \int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)(\beta-\gamma'))\partial_{x_i}\overline{w}_0(x,\theta_j)\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx$$

and

$$Y^i_{j,j'}(\gamma) = \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\omega_{\gamma+(2l'+1)\gamma'} - \overline{\omega}) C^i_{j,j'}(\gamma,\gamma').$$

Then

$$A^i_{j,j'} = \frac{1}{(2l'+1)^d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma). \quad (3.37)$$

Notice that $(Y^i_{j,j'}(\gamma))_{\gamma\in\mathbb{Z}^d_{2l'+1}}$ are bounded random and independent variables with $\mathbb{E}(Y^i_{j,j'}(\gamma)) = 0$. Indeed, using the fact that ρ^0 is compactly supported and (3.30) we get that $|Y^i_{j,j'}(\gamma)|$ is bounded. So, to estimate the probability of $\Omega(n,E,\alpha)$ it suffices to estimate the probability that

$$\frac{1}{(2l'+1)^d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma) \geq E^\alpha/(4n_0)^2. \quad (3.38)$$

This probability is given by the large deviation principle which gives that [2]

$$\mathbb{P}\Bigg(E^\alpha/(4n_0)^2 \leq \frac{1}{(2l'+1)^d} \cdot \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma) \Bigg) \leq \exp(-c(l')^d E^{2\alpha}) \leq \exp(-cE^{-d/2+2d\rho'+2\alpha}).$$

Here we have used the expression of l'. Using the fact that for our choice of ρ' we have $-d/2+2d\rho'+2\alpha<0$, so for some $\tau>0$ and E sufficiently small, one has

$$\mathbb{P}\left(E^{\eta'} \leq \frac{1}{(2l'+1)^d} \cdot \sum_{\gamma \in \mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma)\right) \leq \exp(-E^{-\tau}).$$

As the probability of $\Omega(n,E,\alpha)$ is bounded by the sum over $1 \leq i \leq d$ and $1 \leq j, j' \leq n_0$ of the probability estimate previously, we get the result of the Lemma 3.1 and consequently the proof of Theorem 1.1 is finished. Indeed, for E small enough we have $E - E^{\alpha} < 0$ and $\overline{H}$ has no spectrum below zero. □

Acknowledgements It is a pleasure to thank Frédéric Klopp for interesting discussion concerning this work and Mounir Ben Salah, the organizer of the spectral theory and EDP conference at Kairouan, for the financial support.

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